HW 4: Fourier Series

UCB

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1 Problem 1: Legendre Polynomials

The Legendre Polynomials are an orthonormal basis for $L^2((-1,1))$ that we will not consider in most of the class. However, they are extremely useful in applied mathematics. In this problem, you will note the basic construction and orthogonality.

Part A) We will take it as a given fact that the polynomials 1, x, x^2,... form a basis of $L^2((-1,1))$ (which occurs because continuous functions may be approximated by polynomials, and L^2 functions may be approximated by continuous functions.

In order to construct the Legendre polynomials, you may apply the Gram-Schmidt Process to these polynomials. Using the Gram-Schmidt Process on $\{1, x, x^2, x^3, ...\}$, compute the first 3 polynomials P_0, P_1, and P_2 of degrees 0, 1, and 2 in x.

Part B) The Legendre polynomial P_n satisfies the differential equation

$$[(1-x^2)P_n'(x)]' + n(n+1)P_n(x) = 0$$

Use this relation to show that

$$\left[(1-x^2)(P_m'P_n - P_n'P_m)\right]' + (m-n)(m+n+1)P_mP_n = 0$$

Part C) Use the above relationship to show that P_n is orthogonal to P_m for $m \neq n$.

2 Problem 2: Example Convergences

In class, we computed that for

$$h(x) = \begin{cases} 0 & x \in [-\pi/2, \pi/2] + 2\pi\mathbb{Z} \\ 1 & x \in (\pi/2, 3\pi/2) + 2\pi\mathbb{Z} \end{cases}$$

we obtain the Fourier Series

$$\frac{1}{2} + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{\pi k} \sin(\frac{\pi k}{2}) \cos(kx)$$
$$= \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi(2n-1)} \cos((2n-1)x)$$

Part A) Show directly that this series does not converge uniformly to $h(x)$.
Part B) Show that this series converges pointwise to $h(x)$ for $x \neq \pi/2 + \pi\mathbb{Z}$.

3 Problem 3: Borthwick 8.2

For $x \in (0, \pi)$, let $g(x) = x$. **Part A)** Extend g to an even function on $\mathbb{T}$ and compute the periodic Fourier coefficients $c_k[g]$ (you should arrive at a cosine series).

Part B)

Does the series converge at 0? Use this to show $\displaystyle\sum_{\substack{k\in\mathbb{N}\\ k \text{ odd}}} \frac{1}{k^2} = \frac{\pi^2}{8}$

0.5 cm

Part C) Using any other identity we have shown for Fourier series, show

$$\sum_{\substack{k\in\mathbb{N}\\ k \text{ odd}}} \frac{1}{k^4} = \frac{\pi^4}{96}$$

4 Problem 4: Borthwick 8.5

Suppose $f \in L^2(-\pi, \pi)$ satisfies $\int_{-\pi}^{\pi} x^l f(x)dx = 0$ for all $l \in \mathbb{N}_0$.

Part A) For $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, define $q_{m.k}(z) = \sum_{l=0}^{m} \frac{(-ikx)^l}{l!}$. Show that $\lim_{m\to\infty} q_{m,k}(x) = e^{-ikx}$ uniformly on $[-\pi, \pi]$. You may use the fact that $\sum_0^\infty \frac{n^k}{k!} < \infty$ for any $n > 0$.

Part B) Show that $\langle q_{m,k}, f\rangle = 0$ for all m and k.

Part C) Conclude that $\langle f, e^{-ikx}\rangle = 0$.

Congratulations! You just proved that the monomials $\{1, x, x^2...\}$ from problem 1 are in fact a basis. Now you can show that the Legendre Polynomials are an orthonormal basis (if you so desire). This also tells you that polynomials are dense in $L^2(\mathbb{T})$, so C^∞ functions are too!